Lectures on 8.4. and 10.4.

3.4. **Existence of Brownian motion.** Let us construct Brownian motion W: W is a continuous process with independent increments, which are normally distributed with $W_t - W_s \sim N(0, t-s)$ for $s < t$. The following construction is due to P. Lévy. We will construct Brownian motion on the interval $[0,1]$. We use special partitions, so-called dyadic partitions $\pi_n := \{k2^{-n+1} : k = 0, \dots, 2^{n-1}\}$. In addition we assume that we have at our disposal independent standard normal random variables $\xi_{n,k}$,defined on $(\Omega, F, \mathbb{P})$. We define the the sequence W^n inductively. Put $W^1(t) := t\xi_{1,1}$. When time point is from the set π_1, then $W^1(0) = 0$ and $W^1(1) = \xi_{1,1}$. Clearly W^1 has normally distributed increments with $W^1(t) - W^1(s) \sim N(0, t-s)$ for $s < t \le 1$. Obviously the increments of W^1 are not independent. Assume now that the process W^n is defined; define the next step by $W^{n+1}(t) = W^n(t)$ for $t \in \pi_n$; for $t = k2^{-n} \in \pi_{n+1} \setminus \pi_n$ we put $W^{n+1}(t) = \frac{1}{2}\left(W^n\left(t - 2^{-n}\right) + W^n\left(t + 2^{-n}\right)\right) + 2^{-(n+1)/2}\xi_{n+1,k}$. Now $W^{(n+1)}$ is defined on π_{n+1}. Between the points $t \in \pi_{n+1}$ we define the process $W^{(n+1)}$ by linear interpolation. By construction, the increments have normal distribution, and the paths of $W^{(n+1)}$ are continuous. Clearly

$$\sup_{t\in[0,1]} |W_t^{n+1} - W_t^n| = 2^{-\frac{n+1}{2}} \max_{k:k2^{-n}\in\pi^{n+1}\setminus\pi^n} |\xi_{n+1,k}|.$$

Using the properties of normal distribution and Borel-Cantelli lemma one can show that the telescopic series

$$W_t^1 + \sum_{k=1}^{\infty} \left(W_t^{n+1} - W_t^n\right)$$

converges almost surely. Limit is a continuous function, because the convergence is uniform in $t \in [0,1]$; since everything is normal, the limit has also normal increments. By checking the covariance structure of the limit one can verify that the limit is a Brownian motion.

3.5. Stopping and localization.

3.5.1. *Stopping.* Let $\mathbb{F}$ be a right continuous history and τ is a stopping time.
The stopped σ- algebra is defined by

$$F_\tau = \{A \in \mathcal{A} : A \cap (\tau \le t) \in F_t \ \forall \ t \ge 0\}.$$

It is known that F_τ is a sigma-algebra, and if $\sigma \le \tau$, then $F_\sigma \subset F_\tau$. Put

$$F_{\tau+} = \cap_{\epsilon>0} F_{\tau+\epsilon};$$

clearly $F_\tau \subset F_{\tau+}$, and from the right continuity of the history $\mathbb{F}$ we obtain that $F_{\tau+} = F_\tau$: If $A \in F_{\tau+}$, then $A \cap (\tau + \epsilon \le t) \in F_t$ for all t. This means that $A \cap (\tau \le t - \epsilon) \in F_t$ for all t, and so $A \cap (\tau \le t) \in F_{t+\epsilon}$. This holds for all $\epsilon > 0$ and we get $A \in F_t$.

Theorem 3.5. *Let $\mathbb{F}$ be a history and τ is a $\mathbb{F}$- stopping time. Then there exists a sequence of stopping times τ^n such that τ^n has only finite number of values and $\tau^n \downarrow \tau$ as $n \to \infty$.*

Proof Define $\tau^n = q2^{-n}$, when $(q-1)2^{-n} \leq \tau < q2^{-n}$, for $1 \leq q \leq n2^{-n}$, and $\tau^n = \infty$, when $\tau > n$; clearly $\tau^n \downarrow \tau$ and $\tau^{(n)}$ is a stopping time

$$\{\tau^n \leq t\} = \cup_{q2^{-n} \leq t}\{\tau \leq q2^{-n}\} \in F_t.$$

□

Remark 3.2. *In general it is not possible to approximate stopping times from the left by stopping times.*

Define the stopped process by X^τ: $X_t^\tau = X_{\tau \wedge t} I_{\{\tau > 0\}}$
Let $(X, \mathbb{F})$ be a process with D- paths , τ is a stopping time and τ^n is an approximating sequence. The stopped process

$$X_t^{\tau^n} = \sum_{q_n \leq t} X_{q_n \wedge t} I_{\{\tau^n = q_n\}} + X_t I_{\{\tau^n > t\}}$$

is adapted to F_t, since $X_{q_n \wedge t} I\{\tau^{(n)} = q_n\} \in F_t$ and $X_t I_{\{\tau^n > t\}} \in F_t$. Because X is right-continuous, then $X_t^\tau = \lim_n X_t^{\tau^n}$ and hence X^τ is adapted to $\mathbb{F}$.
Let X be a process with D- paths and τ is a stopping time. Let τ_n be a sequence of stopping times such that $\tau_n \downarrow\rightarrow \tau$, $\tau_n = \frac{k}{2^n}$, if $\frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}$. and it is easy to see that X_{τ_n} is measurable with respect to F_{τ_n} and $F_{\tau_n} \subset F_{\tau + \frac{1}{2^n}}$. Because $X_\tau = \lim_n X_{\tau_n}$ we get that $X_\tau \in F_{\tau + \frac{1}{2^m}}$ for all m. Hence X_τ is measurable with respect to $F_{\tau+} = F_\tau$.
In the following we need the lemma by Komatsu, which is proved in the exercises:

Lemma 3.2. *Let X be a $\mathbb{F}$-adapted integrable process with D-paths. The process X is a martingale if and only if for every bounded stopping time τ*

$$\mathbb{E}X_\tau = \mathbb{E}X_0.$$

Proof The interesting part of the proof is exercise 10.4. problem 5. □

Theorem 3.6. *Let M be a square intergable martingale and τ, σ are bounded stopping times with $\sigma \leq \tau$. Then $\mathbb{E}[M_\tau | F_\sigma] M_\sigma$.*

Proof let $A \in F_\sigma$ and $\eta = \sigma I_A + \tau I_{A^c}$. Because

$$(\eta \leq t) = ((\sigma \leq t) \cap A) \cup ((\tau \leq t) \cap A^c) \in F_t,$$

then η is a bounded stopping time.
Now by lemma 3.2 we have $\mathbb{E}M_0 = \mathbb{E}M_\eta = \mathbb{E}(M_\sigma I_A) + \mathbb{E}(M_\tau I_{A^c})$ and also $\mathbb{E}M_0 = \mathbb{E}M_\tau = \mathbb{E}(M_\tau I_A) + \mathbb{E}(M_\tau I_{A^c})$. This gives that $\mathbb{E}(M_\sigma I_A) = \mathbb{E}(M_\tau I_A)$ and the claim is proved. □.

3.5.2. *Localization.* The process $(M, \mathbb{F})$ is a local martingale, if there exists an increasing sequence τ^n, $\tau^n \to \infty$ of stopping times such, that M^{τ^n} is a martingale, where $M_t^{\tau_n} = M_{t \wedge \tau_n} I_{\{\tau_n > 0\}}$.
If M is in addition continuous, then we can choose the localizing sequence τ^n in such a way that the stopped martingales M^{τ^n} are bounded.
If M is a local martingale and bounded, then DCT implies that it is a true martingale.

3.6. **Continuous square integrable martingales.**

3.6.1. *Continuous processes with bounded variation.* Let A be a continuous function with bounded variation. If π is a partition of the interval $[0,T]$, $\pi = \{t_k : 0 = t_0 < t_1 < \cdots t_n = T\}$, then the Abel summation formula gives

$$A_T^2 = A_0^2 + 2\sum_{k=1}^n A_{t_{k-1}}(A_{t_k} - A_{t_{k-1}}) + \sum_{k=1}^n (A_{t_k} - A_{t_{k-1}})^2. \tag{3.3}$$

Because A has bounded variation the continuity of A implies

$$\sum_{k=1}^n (A_{t_k} - A_{t_{k-1}})^2 \leq \max_k |A_{t_k} - A_{t_{k-1}}| var_T(A) \to 0,$$

as $\pi \to 0$. On the other hand the left hand side of (3.3) is independent of π, and so there exists a limit

$$\int_0^T A_s dA_s = \lim_{|\pi|\to 0} \sum_k A_{t_{k-1}}(A_{t_k} - A_{t_{k-1}}) = \frac{1}{2}(A_T - A_0).$$

Theorem 3.7. *Let $(M, \mathbb{F})$ be a continuous local martingale. Then M has bounded variation if and only if M is a constant.*

Proof If M is a constant, then $\mathcal{V}_T(M) = 0$ for all $T > 0$. Hence M has bounded variation.
Conversely, let M have a bounded variation. Without loss of generality we can assume that $M_0 = 0$ and we will show that the $\mathbb{P}(M_t = 0 \; \forall \; t \geq 0) = 1$. Let us first assume that $M_\infty^* < K$ and $\mathcal{V}_T(M) < K$ for some $K > 0$, when $T > 0$.
because M has bounded variation and continuous paths, then

$$M_T^2 = 2\int_0^T M_s dM_s.$$

On the other hand for the discretized sum we have

$$|\sum_{k=1}^n M_{t_{k-1}}(M_{t_k} - M_{t_{k-1}})| \leq K^2$$

and

$$\begin{aligned} \sum_{k=1}^n \mathbb{E}(M_{t_{k-1}}(M_{t_k} - M_{t_{k-1}}) &= \sum_{k=1}^n \mathbb{E}\left(\mathbb{E}[M_{t_{k-1}}(M_{t_k} - M_{t_{k-1}})|F_{t_k-1}]\right) \\ &= \sum_{k=1}^n \mathbb{E}\left(M_{t_{k-1}}\mathbb{E}[(M_{t_k} - M_{t_{k-1}})|F_{t_k-1}]\right) = 0; \end{aligned}$$

and so by DCT we also have

$$\mathbb{E}M_T^2 = 2\mathbb{E}\int_0^T M_s dM_s = 0.$$

Hence $M_T = 0$, because M is continuous, then we have $\mathbb{P}(M_t = 0 \; \forall \; t \geq 0) = 1$ [this is true for rational t, and by continuity for all t].
Because M is continuous, then also the mapping $t \mapsto \mathcal{V}_t(M)$ is continuous. For $K > 0$ define a stopping time τ_K by

$$\tau_K = \inf\{t \geq 0 : var_t(M) > K\} \wedge \inf\{t \geq 0 : |M_t| > K\}.$$

For the stopped process M^{τ_K} we have that $\mathbb{P}(M_t^{\tau_K} = 0 \; \forall \; t \geq 0) = 1$. On the other hand $\tau_K \to \infty$, as $K \to \infty$, so $M_t = \lim_K M_t^{\tau_K}$. The theorem is proved. □

Prologue. We want to define stochastic integrals with repsect to a continuous martingale M. Let us describe informally the construction of the stochastic integral in the case the continuous martingale is Brownian motion W. Consider first the integral of a process C of the form $C = \alpha 1_{(u,v]}$, where $\alpha \in F_u$ and $|\alpha| \leq 1$. Then the natural definition of the integral is

$$Y_t = \int_0^t C_s dW_s = \alpha \left(W_{t\wedge v} - W_{t\wedge u} \right).$$

It is not difficult to see that Y is a continuous martingale, and the important isometry holds

$$\mathbb{E} \left(\int_0^\infty C_s dW_s \right)^2 = \mathbb{E} \int_0^\infty C_s^2 ds \qquad (3.4)$$

holds:

$$\begin{aligned} \mathbb{E} \left(\int_0^\infty C_s dW_s \right)^2 &= \mathbb{E} \left(\alpha \left(W_v - W_u \right)^2 \right) \\ &= \mathbb{E} \left(\alpha^2 \left(W_v^2 - W_u^2 \right) \right) \\ &= \mathbb{E} \left(\alpha^2 \left(v - u \right) \right) = \mathbb{E} \int_0^\infty C_s^2 ds, \end{aligned}$$

where the first equality is a property of martingales and the second follows from the fact that the process $W_t^2 - t$ is a martingale. To extend this to the construction of stochastic integral with respect to a continuous martingale M we will consider the Hilbert space of continuous martingales and the analog of the property that $W_t^2 - t$ is a martingale for an arbitrary continuous martingale M.

3.7. **The space of continuous martingales $\mathcal{M}^2$.** Let us define the space $\mathcal{M}^2(\mathbb{F}, \mathbb{P})$ of $\mathbb{L}^2$- bounded continuous martingales M:

- $M \in \mathcal{M}^2$ if M has continuous paths , $M_0 = 0$ and $\sup_t \mathbb{E} M_t^2 < \infty$.

If $M \in \mathcal{M}^2$, then the martingale convergence theorem implies that there exists $M_\infty = \lim_t M_t$, where the convergence is in $\mathbb{L}^2$ and almost surely. Moreover, if τ is a finite stopping time, then $M_\tau = \mathbb{E}[M_\infty | F_\tau]$ and $M_\infty \in F_\infty$, where $F_\infty \doteq \sigma(\cup_t F_t)$.
Let us define a norm in the space $\mathcal{M}^2$ by putting $||M||_{\mathcal{M}^2}^2 \doteq \mathbb{E} M_\infty^2$. We get from Doobs $\mathbb{L}^2$- maximal inequality we obtain that $||M_\infty^*||_{L^2(\mathbb{P})} \leq 2||M||_{\mathcal{M}^2}$ [we will use the notation $||Y||_2$ for the norm in L^2; notice that the $||M||_{\mathcal{M}^2}$ norm is a norm for stochastic process, but the $||Y||_2$ is for random variables].

Theorem 3.8. *The space $(\mathcal{M}^2, || \, ||_{\mathcal{M}^2})$ is a complete space.*

Proof Let M^n c-sequence in the space $\mathcal{M}^2$. Then the sequence of random variables M_∞^n is a c-sequence in the space $L^2(\mathbb{P})$. But this space is complete and there exists a random variable $Y = L^2(\mathbb{P}) - \lim_n M_\infty^n$ with $Y \in F_\infty$.

Let us define a square integrable martingale M by $M_t = \mathbb{E}[Y|F_t]$. Because $Y \in F_\infty$, then $M_\infty = Y$. We have that

$$||(M^n - M)^*_\infty||_2 = ||M^n) - M||_{\mathcal{M}^2} \leq 2||M^{(n)}_\infty - M_\infty||_2 \to 0.$$

From this we obtain that there is a subsequence n_j such that

$$(M^{n_j} - M)^*_\infty \to 0$$

from this we in turn get that the paths of M are almost surely continuous: M is a uniform limit of continuous functions and hence continuous. Moreover, $M_0 = 0$, and so $M \in \mathcal{M}^2$. □

3.8. **The angle bracket process $\langle M, M\rangle$ of a continuous martingale M.** In order to find the extension to the fact that $W_t^2 - t$ is a martingale, we need the following definition.

Definition 3.5. *The process $(C, \mathbb{F})$ is a* <u>*simple predictable process*</u>*, if there exists stopping times $0 \leq \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1}\infty$, and random variables $\alpha_k \in F_{\tau_{k-1}}$, $k \geq 1$, such that*

$$C = \sum_k \alpha_k I_{(\tau_{k-1}, \tau_k]}. \tag{3.5}$$

If $(X, \mathbb{F})$ is a stochastic process and $(C, \mathbb{F})$ a simple predictable process, then we define the stochastic integral $C \circ X$ of C with respect to X by

$$(C \circ X)_t = \sum_k \alpha_k (X_{t\wedge\tau_k} - X_{t\wedge\tau_{k-1}}).$$

Theorem 3.9. *Let $(M, \mathbb{F})$ a continuous martingale and C is a simple predictable process with $|C| \leq 1$. Then $(C \circ M)$ is a continuous square integrable martingale and*

$$\mathbb{E}(C \circ M)_t^2 \leq \mathbb{E}M_t^2.$$

Proof The proof is obvious and the details are left for a voluntary exercise. □

Theorem 3.10. *Let $(M, \mathbb{F})$ be a continuous local martingale. Then there exists a continuous and increasing process $\langle M, M\rangle$, $\langle M, M\rangle_0 = 0$ such that $M^2 - \langle M, M\rangle$ is a local martingale.*

Proof We can assume that $M_0 = 0$.
The uniqueness follows from theorem 3.7: Indeed, if $\langle U, U\rangle$ is another increasing and continuous process, $\langle U, U\rangle_0 = 0$, and $M^2 - \langle U, U\rangle$ is a local martingale. Then the process $\langle U, U\rangle - \langle M, M\rangle = M^2 - \langle M, M\rangle - (M^2 - \langle U, U\rangle)$ has bounded variation and also a local martingale, and by theorem 3.7 it is a constant.
To prove the existence of $\langle M, M\rangle$ let us first assume that M is bounded: $M^*_\infty \leq K$.
For fixed $n \geq 1$ define the stopping times τ_k^n recursively: $\tau_0^n = 0$ and when $k \geq 0$, let us define

$$\tau_{k+1}^n \doteq \inf\{t > \tau_k^n : |M_t - M_{\tau_k^n}| = 2^{-n}\}.$$

Because M is continuous, then $\tau_k^n \to \infty$, when $k \to \infty$ and n is fixed. let us define the processes

$$C_t^n = \sum_k M_{\tau_k^n} I_{\{t \in (\tau_k^n, \tau_{k+1}^n]\}} \ , \ Q_t^n = \sum_k \left(M_{t \wedge \tau_{k+1}^n} - M_{t \wedge \tau_k^n} \right)^2 .$$

As before, by Abel summation formula we obtain

$$M_t^2 = 2(C^n \circ M)_t + Q_t^n, \ t \geq 0. \tag{3.6}$$

The integrals are continuous square integrable martingales and so $C^n \circ M \in \mathcal{M}^2$. By definition we have that $|C_t^n - M_t| \leq 2^{-n}$ and we observe that for $m \leq n$ we have

$$||(C^m \circ M) - (C^n \circ M)||_{\mathcal{M}^2} = ||(C^m - C^n) \circ M||_{\mathcal{M}^2} \leq 2^{-m+1} ||M||_{\mathcal{M}^2}.$$

Hence the sequence $C^n \circ M$ is a c-sequence in the space $\mathcal{M}^2$, and hence there exists a continuous square integrable martingale $N \in \mathcal{M}^2$ such that $C^2 \circ M \xrightarrow{\mathcal{M}^2} N$. Hence the process $\langle M, M \rangle \doteq M^2 - 2N$ is continuous process and from the equality (3.6) we obtain that

$$(Q^n - \langle M, M \rangle)_\infty^* = 2(N - C^n \circ M)_\infty^* \xrightarrow{\mathbb{P}} 0.$$

Next we argue that $\langle M, M \rangle$ is continuous process. Let $\Lambda = \{\tau_k^n : n, k \in \mathbb{N}\}$. Because Q^n converges to the limit $\langle M, M \rangle$ uniformly and Q^n is strictly increasing in the set Λ, so also $\langle M, M \rangle$ is strictly increasing on the set Λ. Moreover, since the process $\langle M, M \rangle$ is continuous, then it increasing also in the closure $\bar{\Lambda}$ of the set Λ. Finally, let $I \subset \bar{\Lambda}^c$ be an interval. From the definition of the set Λ it follows that the martingale M is a constant on I, and hence also Q^n is a constant on this interval, and because $\langle M, M \rangle$ is a uniform limit of Q^n it is also constant on this interval.

Let τ^m be the localizing sequence of M, and let $\sigma_n = \inf\{t \geq 0 : |M_t| = n\}$. Let us define $\eta_n = \tau^n \wedge \sigma_n$. We have that $\eta_n \to \infty$ as $n \to \infty$. Since the process M^{η_n} is a bounded continuous martingale, we know by the proof above that there is a unique process $\langle M^{\eta_n}, M^{\eta_n} \rangle$. If $m < n$, then $\eta_m < \eta_n$ and from the equality $(M^{\eta_n})^{\eta_m} = M^{\eta_m}$ we obtain that $\langle M^{\eta_m}, M^{\eta_m} \rangle = \langle M^{\eta_n}, M^{\eta_n} \rangle^{\eta_m}$.

Hence on the random interval $[0, \eta_m]$ $\langle M^{\eta_m}, M^{\eta_m} \langle = \langle M^{\eta_n}, M^{\eta_n} \rangle$ almost surely. Because $\eta_n \uparrow \infty$,we finally obtain that there exists

$$\langle M, M \rangle = \lim_n \langle M^{\sigma_n}, M^{\sigma_n} \rangle$$

and $\langle M, M \rangle$ is continuous, increasing, and adapted to $\mathbb{F}$. Further $\langle M, M \rangle = \langle M^{\eta_n}, M^{\eta_n} \rangle$ on the interval $[0, \eta_n]$ and $(M^{\eta_n})^2 - \langle M^{\eta_n}, M^{\eta_n} \rangle$ is a martingale for every n, and so $M^2 - \langle M, M \rangle$ is a local martingale. □

Corollary 3.1. *Let* $(M, \mathbb{F})$ *and* $(N, \mathbb{F})$ *be two continuous local martingales. Then there is a unique* $\mathbb{F}$*- adapted process* $\langle M, N \rangle$ *such that* $MN - \langle M, N \rangle$ *is a local martingale.*

Proof We have the polarization identity

$$MN = \frac{1}{4} \left((M + N)^2 - (M - N)^2 \right);$$

then by defining $\langle M.N \rangle = \frac{1}{4}(\langle M + N, M + N \rangle - \langle M - N, M - N \rangle)$, we get that the process $MN - \langle M, N \rangle$ is a local martingale. The process $\langle M, N \rangle$

has bounded variation, since it is a difference of two increasing processes. . Uniqueness can be proved using again 3.10. □

Remark 3.3. *We know that Brownian motion satisfies:*

$$\sum_{t_k \in \pi} (W_{t_k} - W_{t_{k-1}})^2 \xrightarrow{\mathbb{P}} T, \ |\pi| \to 0$$

where $\pi = \pi^n$ *is a partition of the interval* $[0, T]$ *. This is true fro a continuous local martingale* M *:*

$$\sum_{t_k \in \pi} (M_{t_k} - M_{t_{k-1}})^2 \xrightarrow{\mathbb{P}} \langle M, M \rangle_T;$$

in the proof of theorem 3.9 we have proved this over special random partitions, but one can show that this is true for deterministic partitions as well.

4. Stochastic integrals

4.1. Stochastic integral of a simple predictable process.

History. If $f \in C_0^\infty$, then stochastic integral [of the deterministic process f] with respect to Brownian motion W can be defined by the integration by parts formula

$$\int_0^T f_s dW_s = f_T W_T - f_0 W_0 - \int_0^T W_s df_s;$$

this was the approach of Norbert Wiener, and for this reason such integrals, where the integrand is a deterministic function are called Wiener integrals. The stochastic integral with respect to Brownian motion appeared in the work of Japanese mathematician Kyoshi It in the 1940's, and independently also in the works of Ukrainian mathematician Iosif Gihman at the same time.

4.1.1. *Measurability concepts.* Define the following σ- algebras on the product space $\Omega \times \mathbb{R}_+$:

- Predictable σ- algebra $\mathcal{P}(\mathbb{F})$ is the smallest σ- algebra, which makes all $\mathbb{F}$ adapted, left-continuous processes measurable, in other words
$$\mathcal{P}(\mathbb{F}) = \sigma\{X \in \mathbb{F} : X \quad \text{is left-continuous}\}.$$
- Optional σ- algebra $\mathcal{O}(\mathbb{F})$ is the smallest σ- algebra, which makes right-continuous $\mathbb{F}$-adapted processes measurable, in other words
$$\mathcal{O}(\mathbb{F}) = \sigma\{X \in \mathbb{F} : X \quad \text{is right-continuous}\}.$$
- The process X is progressively measurable, if the mapping $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ restricted to the interval $[0, t]$ is $F_t \otimes \mathbb{B}_{[0,t]}$- measurable. The progressive σ- algebra $Prog(\mathbb{F})$ is generated by the progressively measurable processes.

Remark 4.1. *One can show that*

$$\mathcal{P}(\mathbb{F}) \subset \mathcal{O}(\mathbb{F}) \subset Prog(\mathbb{F}).$$

4.2. **Stochastic integral.** We want to define stochastic integral of a predictable process H with respect to a continuous square integrable local martingale M when

$$\mathbb{E}\int_0^\infty H_u^2 d\langle M, M\rangle_u < \infty; \tag{4.1}$$

here the integral with respect to the increasing process $\langle M, M\rangle$ is a Riemann-Stieltjes- integral:

$$\int_0^\infty H_s d\langle M, M\rangle_s = \lim_{|\pi|\to 0} \sum_k H_{t_{k-1}} \left(\langle M, M\rangle_{t_k} - \langle M, M\rangle_{t_{k-1}}\right)$$

and π is a partition of the interval $[0, \infty)$.

We start with the most simple case with H a left-continuous process: $H = \alpha I_{(a,b]}$, $0 \le a < b < \infty$. In order H to be adapted to $\mathbb{F}$, we must assume that $\alpha \in F_{a+} = F_a$. Define the stochastic integral by putting

$$(H \circ M)_t = \alpha(M_{t\wedge b} - M_{t\wedge a}).$$

Lemma 4.1. *Assume that $(M, \mathbb{F})$ is a continuous martingale with $M \in \mathcal{M}_2$. Let $H = \alpha I_{(a,b]}$ and put $N \doteq (H \circ M)$. If $\alpha \in F_a$ and α is bounded, then N is a continuous martingale and we have the isometry*

$$\mathbb{E}N_\infty^2 = \mathbb{E}[\alpha^2 \left(\langle M, M\rangle_b - \langle M, M\rangle_a\right)] \tag{4.2}$$

and

$$\langle N, N\rangle_t = \int_0^t \alpha^2 I_{(a,b]}(u) d\langle M, M\rangle_u. \tag{4.3}$$

Proof Clearly N is continuous, adapted to $\mathbb{F}$ and integrable. Let us show the martingale property of N.

If $a < s < t < b$, then $N_t = \alpha(M_t - M_a)$ and

$$\mathbb{E}[\alpha(M_t - M_a)|F_s = \alpha\mathbb{E}[(M_t - M_a)|F_s] = \alpha(M_s - M_a) = N_s.$$

The rest of the cases with ab, s, t are proved similarly.

The proof of (4.2) and (4.3) is an exercise

□